Revenue maximization with second-order conditions

If the joint production function is $x = q_1^2 + q_2^2$ and the selling prices are $p_1 = 20$ and $p_2 = 10$. Find the maximum revenue if the amount of input must be $x = 100$. Verify the second-order conditions.

Solution

We construct the Lagrangian:

$$L = 20q_1 + 10q_2 + \lambda(100 - q_1^2 - q_2^2)$$

$$L'q_1 = 20 - \lambda 2q_1 = 0$$

$$L'q_2 = 10 - \lambda 2q_2 = 0$$

$$L'\lambda = 100 - q_1^2 - q_2^2 = 0$$

We solve for λ and set them equal:

$$\frac{20}{2q_1} = \frac{10}{2q_2}$$

$$10q_2 = 5q_1$$

$$2q_2 = q_1$$

Insert into the third condition:

$$100 - (2q_2)^2 - q_2^2 = 0$$

$$100 - 4q_2^2 - q_2^2 = 0$$

$$20 = q_2^2$$

$$q_2 = \sqrt{20}$$

$$q_1 = 2\sqrt{20}$$

Additionally, the value of $\lambda = \frac{10}{2q_2} = \frac{5}{\sqrt{20}}$ For the second-order conditions, we calculate the bordered Hessian:

$$|H| = \begin{vmatrix} 0 & 2q_1 & 2q_2 \\ 2q_1 & -2\lambda & 0 \\ 2q_2 & 0 & -2\lambda \end{vmatrix} = \begin{vmatrix} 0 & 4\sqrt{20} & 2\sqrt{20} \\ 4\sqrt{20} & -2\frac{5}{\sqrt{20}} & 0 \\ 2\sqrt{20} & 0 & -2\frac{5}{\sqrt{20}} \end{vmatrix}$$

Calculating the determinant:

$$-4\sqrt{20}(-40) + 2\sqrt{20}(20) = 894.4272 > 0$$

Therefore, it is a constrained maximum.